APPENDIX B

STIFFNESS OR DISPLACEMENT METHOD

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APPENDIX B

STIFFNESS OR DISPLACEMENT METHOD

B.1 Introduction:

I. Brief Discussion of Force or Flexibility Method

Indeterminate systems comprise the large majority of structures to be analyzed and designed and hence the solution process must satisfy the conditions of compatibility and the material stress-strain behavior. Traditional methods of structural analysis employing the concept of redundancies and "consistent deformations" have not proven to be as simple and direct in application as the "stiffness" or "displacement" approach to be treated here and also used in the STRUDL program. The traditional method involving redundancies has been formalized into a matrix approach and is now referred to as the "force" method.

a. Propped Cantilever Example: The "force method" of structural analysis (often referred to as the "flexibility method") is probably most familiar to us for the solution of statically indeterminate structures. The propped cantilever beam of Figure B.1a provides a simple example of the use of the force method.

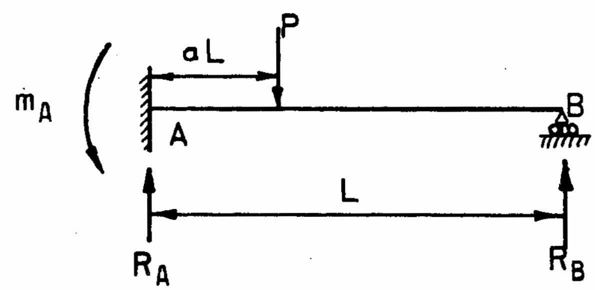

Fig. B.1a

A concentrated load, P, is acting at a distance aL from the left support. This load produces the reactions R_A, M_A and R_B as shown. Since we have only two equations of equilibrium, $\Sigma F_v = 0$ and $\Sigma M = 0$, and three unknown reactions, this beam is considered to be indeterminate to the first degree. To gain an additional equation we consider the deflections of the structure. The traditional way to approach this problem is to remove one of the redundant reactions, in this case R_B, and determine the deflection, δ_o at B on the statically determinate cantilever due to the external load P, Figure B.1b. Since our actual structure does not have a vertical deflection at B the redundant reaction, R_B, must be of such

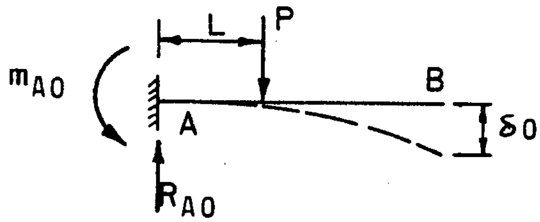

Fig. B.1b

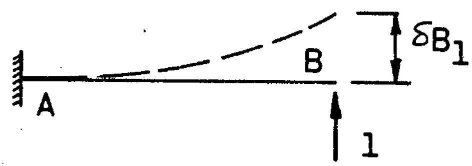

Fig. B.1c

a magnitude that it pushes the beam of Figure B.1b upward with a displacement equal to δ_o. If we apply a unit value of the redundant R_B to the cantilever shown in Figure B.1c, we will have a deflection at B upward equal to δ_o. Therefore we can write

$$\delta_0 + R_B \delta_{B1} = 0 \qquad (1)$$

This is our compatibility equation saying that the deflection at B is zero. Here δ_{B1} is the vertical deflection at B due to a unit load at B. We solve Eq. 1 for R_B.

$$R_B = \frac{\delta_o}{\delta_{B1}} \qquad (2)$$

Having R_B allows us to determine M_A and R_A by statics.

b. Four Span Beam Example: For a beam with a larger number of redundancies we could proceed in a very similar manner. For example, consider the four span beam of Figure B.1d. In this case we can consider R_1, R_2, R_3 and R_4 as the redundants, leaving us with the cantilever beam of Figure B.1e.

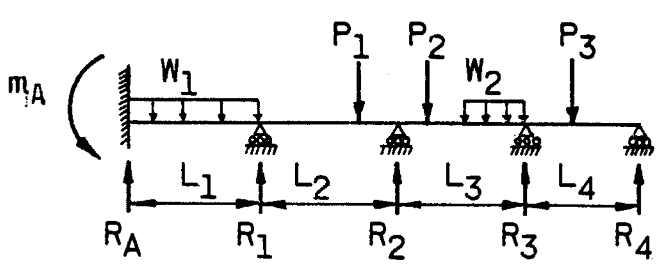

Fig. B.1d

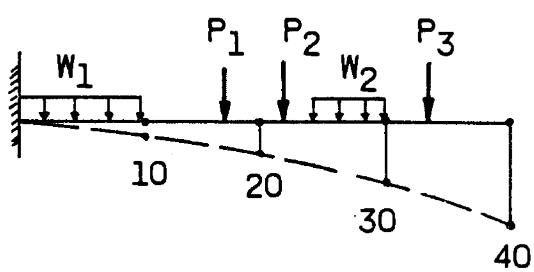

Fig. B.1e

The applied loads produce deflections $\delta_{10}, \delta_{20}, \delta_{30}$ and δ_{40}. As before, these deflections do not represent the true state of our structure so we must consider that the redundants push upward just enough to eliminate these displacements. In this instance we shall arrive at four compatibility conditions. For example, applying a unit load at support 1 yields deflections δ_{11}, δ_{21}, δ_{31} and δ_{41} (see Figure B.1f). Similarly, a unit load at point 2 yields δ_{12}, δ_{22}, δ_{32}, δ_{42}.

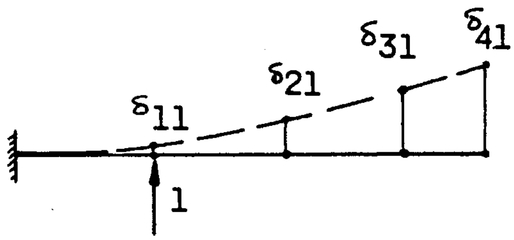

Fig. B.1f

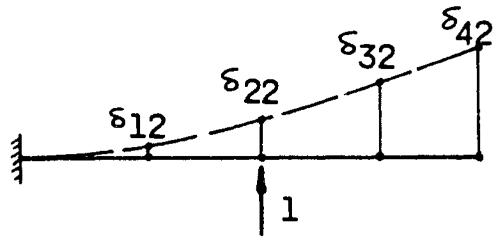

Fig. B.1g

We could continue applying the unit load at each point and determining the deflections. Our compatibility equations become

$$
\begin{aligned}
\delta_{10} + R_1\,\delta_{11} + R_2\,\delta_{12} + R_3\,\delta_{13} + R_4\,\delta_{14} &= 0 \\
\delta_{20} + R_1\,\delta_{21} + R_2\,\delta_{22} + R_3\,\delta_{23} + R_4\,\delta_{24} &= 0 \\
\delta_{30} + R_1\,\delta_{31} + R_2\,\delta_{32} + R_3\,\delta_{33} + R_4\,\delta_{34} &= 0 \\
\delta_{40} + R_1\,\delta_{41} + R_2\,\delta_{42} + R_3\,\delta_{43} + R_4\,\delta_{44} &= 0
\end{aligned}
\qquad (3)
$$

Note that δ_{ij} is the deflection at support i due to a unit load at support j. The solution of these four equations gives values for the redundants R_1, R_2, R_3 and R_4. One thing that we can observe is that in order to determine the redundants we must calculate the deflections at <u>all</u> of the redundant points for <u>all</u> positions of the unit load.

II. Brief Discussion of the Stiffness or Displacement Methods

The previous discussion has dealt with the flexibility or force method of analysis. We can handle the same problems by considering the stiffness or displacement method of analysis. In this case we take the unknown displacements of the structure as the redundants.

a. Propped Cantilever Example: Again consider the propped cantilever beam of Figure B.1h.

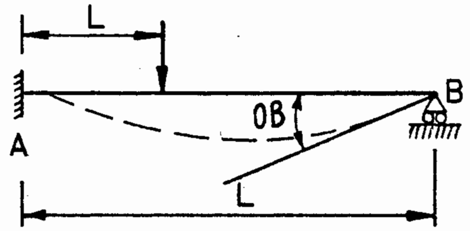

Fig. B.1h

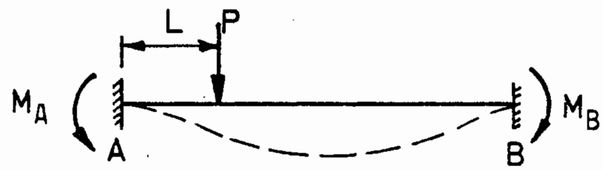

Fig B.1c

In this case the only unknown displacement is the rotation, θ_B, at end B. We then say that this structure only has one degree of freedom. In order to eliminate this unknown displacement we clamp the end. The applied loads then produce fixed end moments M_A and M_B as shown in Figure B.1i. However, we know that this is not the actual condition of our structure. The redundant rotation, θ_B, produces a moment of magnitude equal to M_B but of opposite direction. We can consider the effect of this rotation by considering the effect of a unit rotation at end B (Figure B.1j).

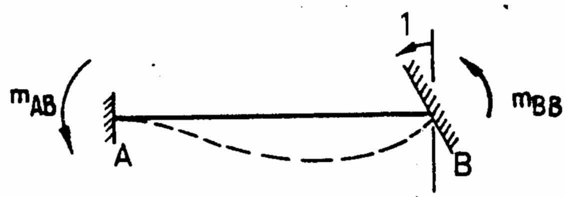

Fig. B.1j

Then our compatibility condition becomes

$$M_B + m_{BB}\,\theta_B = 0 \qquad (3)$$

$$\theta_B = \frac{M_B}{m_B} \qquad (4)$$

where m_{BB} is the moment at B due to a unit rotation at B. Having θB, we can determine all other moments and reactions. For example, the moment at A, M_A, is given by

$$M_A = M_A^{FEM} + m_{AB}\,\theta_B \qquad (5)$$

where m_{AB} is the moment at A due to a unit rotation at B.

b. <u>Four Span Beam Example</u>: This may be a strange way to look at this problem because it is normally more difficult to calculate the reaction caused by a unit displacement than to calculate the displacement caused by a unit reaction. However, we shall soon see that there is an advantage in looking at the problem from this point of view. Consider again the continuous beam of Figure B.1k. We see that there are four unknown rotations, $\theta_1, \theta_2, \theta_3, \theta_4$

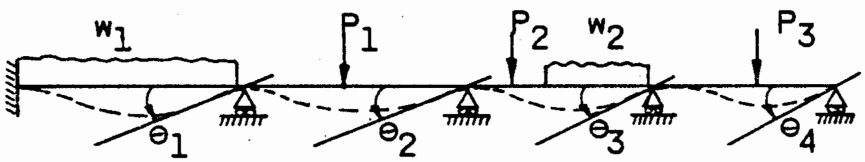

Fig. B.1k

We begin by fixing all supports against rotation and determine the FEM's (Figure B.1L).

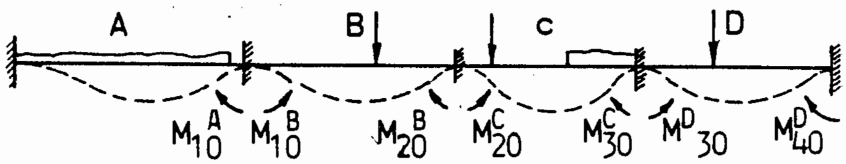

Fig. B.1L

Now we apply a unit rotation at each support. For example, a unit rotation at support 1 (Figure B.1m) produces moments M_{L1} at the left support, m_{11} and m_{21} at the first and second supports respectively.

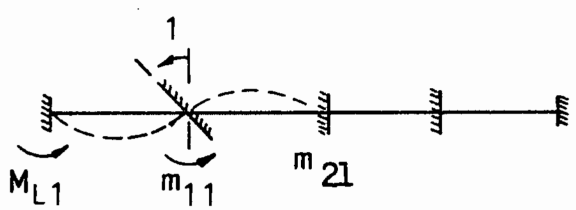

Fig. B.1m

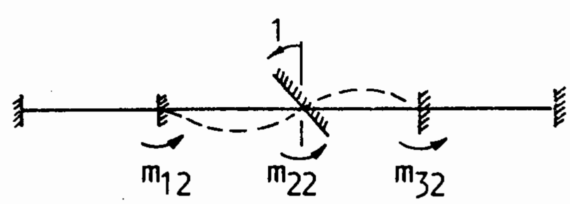

Fig. B.1n

A unit rotation at support 2 yields (Figure B.1n) moments m_{12}, m_{22} and m_{32} at supports 1, 2 and 3, respectively. We can write the compatibility equation at support 1.

$$M^A_{10} + M^B_{10} + m_{11}\,\theta_1 + M_{12}\,\theta_2 = 0 \qquad (6)$$

We can continue to apply the unit rotation and get three additional compatibility equations, for example, at joint 2

$$0 = M^C_{20} + M^B_{20} + m_{21}\,\theta_1 + m_{22}\,\theta_2 + m_{23}\,\theta_3 \qquad (7)$$

Solving this system of equations gives values for θ_1, θ_2, θ_3 and θ_4 . The thing to note is that these equations only involve the effects produced by members adjacent to the joint in question. In other words we do not have to determine effects on the structure due to rotations at distant points. This allows an efficient manner of storing the problem in the computer and allows the computer time to be reduced. Furthermore, the method is applicable to determinate and indeterminate systems with equal ease and the degree of indeterminancy need not even be determined.

The preceding discussion of stiffness method was presented to give an overview of the method. We shall next consider a more detailed application of the stiffness or displacement approach.

B.2 Basic Displacement Approach Using Example Problem 2.3, Indeterminate Truss

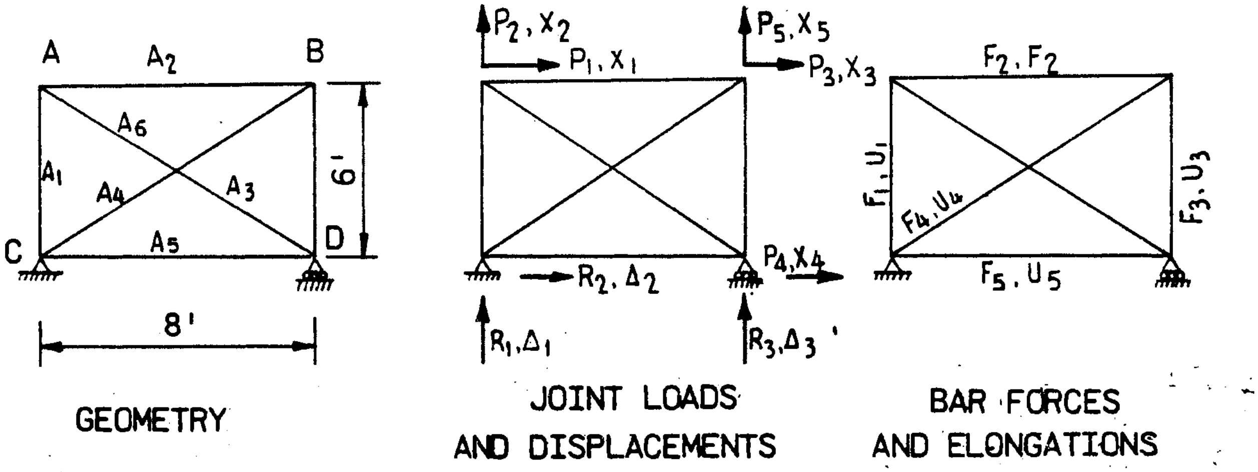

fig. B.2a

Note that the external applied loads, P, have a one-to-one correspondence with the external joint displacements, X, and the internal bar forces, F, have a one-to-one correspondence with the member elongations, u. Also note the one-to-one correspondence between unknown reaction components, R and known support displacements Δ (not necessarily zero).

a. Equilibrium Matrix: Rewriting the equilibrium matrix, shown on Page A-10, to include the additional bar gives

$$\begin{bmatrix} 3. & 0. \\ -4. & -10. \\ 0. & -4. \\ 0. & 0. \\ -10 & 0. \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & -.8 \\ 1. & 0 & 0 & 0 & 0 & .6 \\ 0 & 1 & 0 & .8 & 0 & .0 \\ 0 & 0 & 0 & 0 & 1 & .8 \\ 0 & 0 & 1 & .6 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \\ F_{41} & F_{42} \\ F_{51} & F_{52} \\ F_{61} & F_{62} \end{bmatrix} \qquad \{P\} = [A]\,\{F\} \qquad (8)$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -.6 & 0 & 0 \\ 0 & 0 & 0 & -.8 & -1 & 0 \\ 0 & 0 & .1 & 0 & 0 & -.6 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \\ F_{41} & F_{42} \\ F_{51} & F_{52} \\ F_{61} & F_{62} \end{bmatrix} \qquad \{R\}\,[A_R]\,\{F\}$$

b. <u>Compatibility Matrix</u>: The equilibrium matrix is no longer square (A_{5x6}) and hence cannot be inverted. The degree of indeterminacy is NF - NP = 6 - 5 = 1 Therefore, we must apply the stress-strain relationships and the conditions of compatibility.

The stress-strain assumption for axially loaded members is simply

$$\sigma_x = E\epsilon_x \text{ or } \frac{F_x}{A_x} = E\frac{U_x}{L} \text{ or } F_x = \frac{EA_x}{L} U_x \qquad (9)$$

One such equation can be written for each bar, hence

$$\{F\}_{6x2} = [S]_{6x6} \{u\}_{6x2}$$

$$\{F\} = \begin{bmatrix} EA_1/L_1 & & & & & \\ & EA_2/L_2 & & & \text{ZEROS} & \\ & & EA_3/L_3 & & & \\ & & & EA_4/L_4 & & \\ & \text{ZEROS} & & & EA_5/L_5 & \\ & & & & & EA_6/L_6 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \\ u_{41} & u_{42} \\ u_{51} & u_{52} \\ u_{61} & u_{62} \end{bmatrix} \qquad (10)$$

Applying compatibility conditions to a truss means simply that the member elongations $\{u\}$ must be consistent with the joint displacements $\{x\}$ and $\{\Delta\}$.

$$\begin{aligned} u_1 &= b_{11}x_1 + b_{12}x_2 + \cdots + b_{15}x_5 + b_{16}\Delta_1 + \cdots + b_{18}\Delta_3 \\ u_2 &= b_{21}x_1 + b_{22}x_2 + \cdots + b_{25}x_5 + b_{26}\Delta_1 \quad \cdot \\ u_3 &= b_{31}x_1 \quad \cdot \quad \cdot + b_{36}\Delta_1 \quad \cdot \\ u_4 &= b_{41}x_1 \quad \cdot \quad \cdot + b_{46}\Delta_1 \quad \cdot \\ u_5 &= b_{51}x_1 \quad \cdot \quad \cdot + b_{56}\Delta_1 \quad \cdot \\ u_6 &= b_{61}x_1 + b_{62}x_2 + \cdots + b_{65}x_5 + b_{66}\Delta 1 + \cdots + b_{68}\Delta_3 \end{aligned} \qquad (11)$$

$$\{u\} = [B] \{x\} + [B_R]\{\Delta\} \qquad (12)$$

What we are saying is that bar elongations are some linear combination of the external joint displacements.

To obtain the coefficients of [B] defining the compatibility matrix, we may apply a unit displacement in the direction of each of the external joint displacements. For example,

$$X_1 = 1;\ X_2 = X_3 = X_4 = X_5 = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

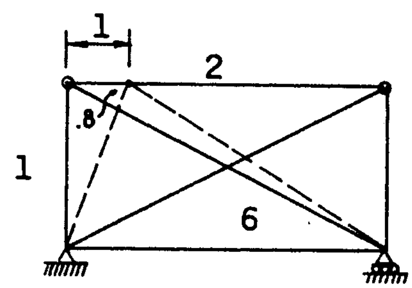

Fig. B.2b

$u_1 = b_{11},\ u_2 = b_{21},\ u_3 = b_{31},$ etc

$u_1 = b_{11} = 0$ (small deflections)

$u_2 = b_{21} = -1.0$

$u_6 = b_{61} = -0.8$

$u_3 = u_4 = u_5 = 0$

as another example set $X_5 = 1;\ X_1 = X_2 = X_3 = X_4 = \Delta_1 = \Delta_2 = \Delta_3 = 0$

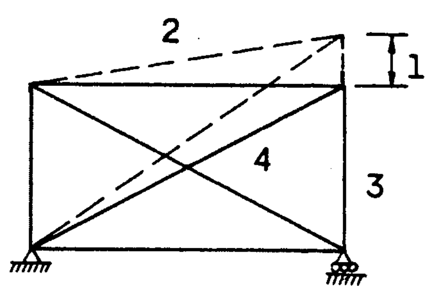

Fig. B.2c

$u_2 = b_{25} = 0$ (small deflections)

$u_3 = b_{35} = +1.$

$u_4 = b_{45} = +0.6$

$u_1 = u_5 = u_6 = 0.$

Each column may be determined successively to yield

$$[B]=\begin{bmatrix} 0 & 1. & 0 & 0 & 0 \\ -1. & 0 & 1. & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & .8 & 0 & .6 \\ 0 & 0 & 0 & 1. & 0 \\ -.8 & .6 & 0 & .8 & 0 \end{bmatrix} \quad [B_R]=\begin{bmatrix} -1. & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1. \\ -.6 & -.8 & 0 \\ 0 & -1. & 0 \\ 0 & 0 & -.6 \end{bmatrix} \qquad (13)$$

(we've just done columns 1 and 5)

c. <u>Relationship of Compatibility and Equilibrium Matrices</u>: It is extremely interesting and significant to note at this time the transpose relationship between the equilibrium and the compatibility matrices or

$$[B] = [A]^T$$
$$[B_R] = [A_R]^T \quad (14)$$

This relationship always holds for linearly elastic structures and can be proved by the principle of virtual work.

d. <u>System Stiffness Matrix</u>: In summary

$$\{P\} = [A]\{F\}$$
$$\{R\} = [A_R]\{F\} \qquad \text{Equilibrium} \qquad (15)$$
$$\{F\} = [S]\{u\} \qquad \text{Stress-Strain} \qquad (16)$$
$$\{u\} = [A]^T\{x\} + [A_R]^T\{\Delta\} \qquad \text{Compatibility} \qquad (17)$$

Substitute (17) into (16) to obtain

$$\{F\} = [SA^T]\{x\} + [SA_R^T]\{\Delta\} \qquad (18)$$

Then substitute 18 into 15

$$\{P\} = [ASA^T]\ \{x\} + [ASA_R^T]\,\{\Delta\} \tag{19}$$

$$\{P\} - [ASA_R^T]\{\Delta\} = [K]\ \{x\}$$

$$\{x\} = [K]^{-1}\ [\ \{P\}\ - [ASA_R^T]\,\{\Delta\}\ ] \tag{20}$$

$$\{R_1\} = [A_R S A^T]\ \{x\} + [A_R S A_R^T\ ]\ \{\Delta\} \tag{21}$$

If all support displacements are zero the basic solution process for the example is

$$\{x\}_{5\times2} = [ASA^T]^{-1}_{5\times5}\ \ \{P\}_{5\times2} = [K]^{-1}_{5\times5}\ \{P\}_{5\times2} \tag{22}$$

$$\{F\}_{6\times2} = [SA^T]_{6\times5}\ \{x\}_{5\times2} \tag{23}$$

$$[SA^T] = \begin{bmatrix} 0 & \frac{EA_1}{L_1} & 0 & 0 & 0 \\ -EA_2 & 0 & EA_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{EA_3}{L_3} \\ 0 & 0 & \frac{.8EA_4}{L_4} & 0 & \frac{.6EA_4}{L_4} \\ 0 & 0 & 0 & \frac{EA_5}{L_5} & 0 \\ \frac{-.8EA_6}{L_6} & \frac{.6EA_6}{L_6} & 0 & \frac{.8EA_6}{L_6} & 0 \end{bmatrix} \tag{24}$$

and

$$[K] = (E)\begin{bmatrix} \left(\frac{A_2}{L_2} + .64\frac{A_6}{L_6}\right) & -.48\frac{A_6}{L_6} & -\frac{A_2}{L_2} & -.64\frac{A_6}{L_6} & 0 \\ -.48\frac{A_6}{L_6} & \left(\frac{A_1}{L_1} + .36\frac{A_6}{L_6}\right) & 0 & -.48\frac{A_6}{L_6} & \\ \frac{A_2}{L_2} & 0 & \left(\frac{A_2}{L_2} + .64\frac{A_6}{L_6}\right) & 0 & .48\frac{A_6}{L_6} \\ -.64\frac{A_6}{L_6} & -48\frac{A_6}{L_6} & 0 & \left(\frac{A_5}{L_5} + .69\frac{A_6}{L_6}\right) & 0 \\ 6 & 0 & .48\frac{A_4}{L_4} & 0 & \left(\frac{A_3}{L_3} + .36\frac{A_4}{L_2}\right) \end{bmatrix} \quad (25)$$

These two matrices plus the load matrix $\{P\}$ are what is required to solve for the displacements and forces in the structural system under investigation. The method is generally referred to as the displacement method because displacements are the primary unknown quantities. Also note the symmetrical condition of the stiffness matrix. This is proved by the reciprocity theorem.

B.3 "Direct Stiffness" Approach Using Example Problem 2.3, Indeterminate Truss

Direct stiffness simply implies that one is going to obtain the stiffness matrix [K] without generating the [A] [S] , and [B] matrices and then performing the matrix multiplication operations. This method is much more efficient computationally and requires considerably less effort in the preparation of data input.

a. Discussion of the development of the system stiffness matrix directly from physical considerations.

To motivate the development consider the indeterminate truss just investigated and write the basic stiffness equations as

$$\begin{aligned}
P_1 &= K_{11}X_1 + K_{12}X_2 + K_{13}X_3 + K_{14}X_4 + K_{15}X_5 \\
P_2 &= K_{21}X_1 + K_{22}X_2 + K_{23}X_3 + K_{24}X_4 + K_{25}X_5 \\
P_3 &= K_{31}X_1 + K_{32}X_2 + K_{33}X_3 + K_{34}X_4 + K_{35}X_5 \\
P_4 &= K_{41}X_1 + K_{42}X_2 + K_{43}X_3 + K_{44}X_4 + K_{45}X_5 \\
P_5 &= K_{51}X_1 + K_{52}X_2 + K_{53}X_3 + K_{54}X_4 + K_{55}X_5
\end{aligned} \tag{26}$$

Again these coefficients may be determined by defining a set of values for the independent variables $\{X\}$, in order to isolate one column of the matrix. For example, if $X_1 = 1$; $X_2 = X_3 = X_4 = X_5 = 0$, then $P_1 = K_{11}$, $P_2 = K_{21}$, $P_3 = K_{31}$, $P_4 = K_{41}$, and $P_5 = K_{51}$. Physically this means that for a given state of displacement, what are the required applied loads to produce this state? Therefore, the stiffness coefficient K_{ij} is defined to be the load at coordinate i given a unit displacement at coordinate j, all other displacements equal to zero.

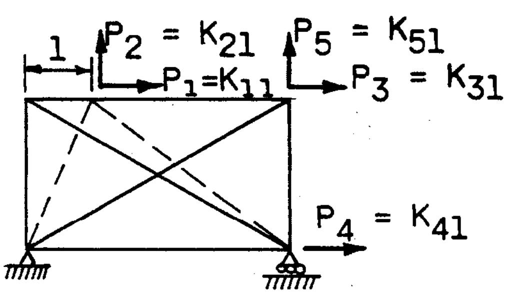

Fig. B.3a

For this state of displacements the member elongations are:

$u_1 = 0$; $u_2 = -1$; $u_3 = 0$;

$u_4 = 0$; $u_5 = 0$; $u_6 = -0.8$

Hence the associated bar forces are

$$F_1 = 0;\ F_2 = (-1)\frac{EA_2}{L_2};\ F_3 = 0;\ F_4 = 0;\ F_5 = 0;\ F_6 = -0.8\frac{EA_6}{L_6} \qquad 27$$

Then consider the equilibrium of the joints

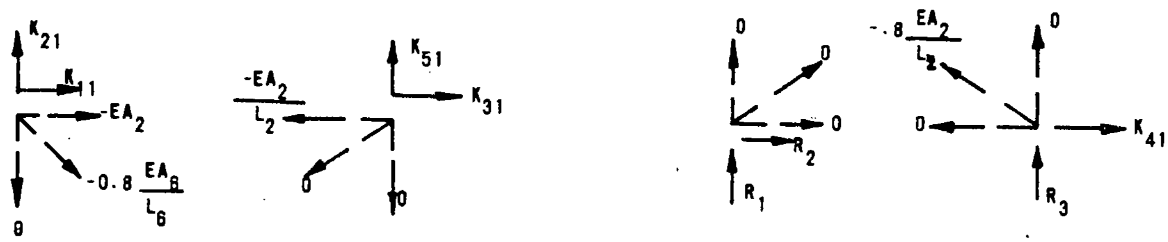

Fig. B.3b

$$\begin{aligned}
&K_{11} + \left(-\frac{EA_2}{L_2}\right) + 0.8\left(-.8\frac{EA_6}{L_6}\right) = 0 && K_{11} = \frac{EA_2}{L_2} + .64\frac{EA_6}{L_6} \\
&K_{21} - 0 -.0.6\left(-0.8\frac{EA_6}{L_6}\right) = 0 && K_{21} = -.48\frac{EA_6}{L_6} \\
&K_{31} - \left(-\frac{EA_2}{L_2}\right) - 0.8\,(0) = 0 && K_{31} = -\frac{EA_2}{L_2} \qquad (28) \\
&K_{41} - 0 - 0.8\left(-0.8\frac{EA_2}{L_2}\right) = 0 && K_{41} = -.64\frac{EA_2}{L_2} \\
&K_{51} - 0 - 0.6\,(0) = 0 && K_{51} = 0
\end{aligned}$$

These coefficients are the same as those obtained in the first column of the [K] matrix when the triple matrix multiplication was employed. If a similar operation is employed for each of the external displacement coordinates the remaining four columns of the stiffness matrix could be obtained and would agree with those obtained previously.

Using this concept to develop the stiffness matrix indicates the composition of the individual terms and also clearly identifies which members of the system will contribute to the individual coefficients. In particular, any given member will only contribute to those coefficients associated with the external coordinates of the ends of the member. The coefficients K_{ii} will consist of contributions from each member framing into the joint associated with coordinate i. In more general terms each member contributes to the stiffness of the joints into which they frame. This suggests that the stiffness matrix could be generated from the stiffness properties of the component parts or as the summation of the element stiffness matrices.

First take an individual truss bar, subjected to an axial force, F.

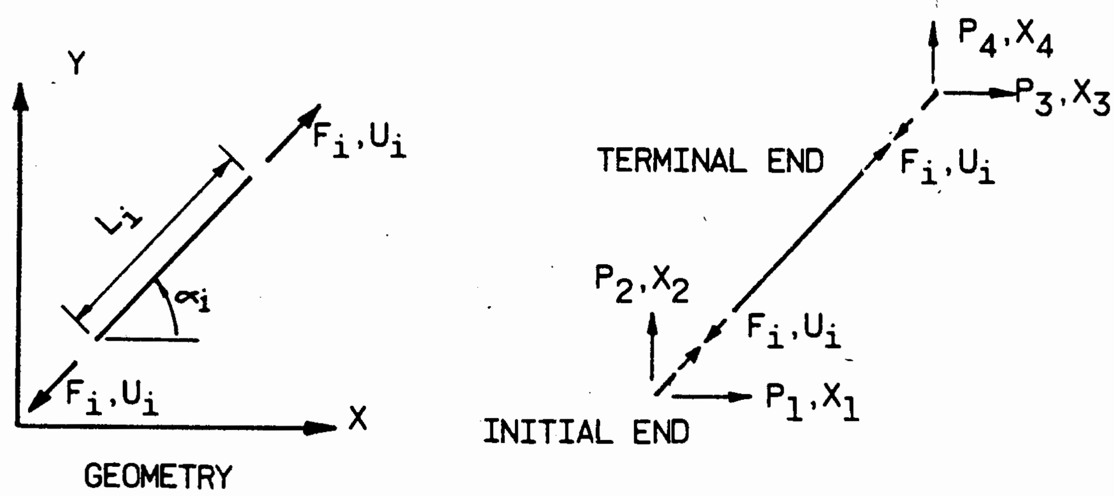

Fig. B.3c

EQUILIBRIUM STRESS-STRAIN COMPATIBILITY

$$P_1 = -F_i \cos\alpha_i$$
$$P_2 = -F_i \sin\alpha_i$$
$$P_3 = +F_i \cos\alpha_i$$
$$P_4 = +F_i \sin\alpha_i$$

$$F_i = \frac{EA_i}{L_i}$$

$$U_i = -X_1 \cos\alpha_i - X_2 \sin\alpha_i + X_3 \cos\alpha_i + X_4 \sin\alpha_i$$

Now the element stiffness matrix [EK] can be determined as the product of the element equilibrium, stress-strain, and compatibility matrices, or

$$[EK] = \begin{bmatrix} -\cos\alpha_i \\ -\sin\alpha_i \\ \cos\alpha_i \\ \sin\alpha_i \end{bmatrix} \begin{bmatrix} \frac{EAi}{Li} \end{bmatrix} \begin{bmatrix} -\cos\alpha_i & -\sin\alpha_i & \cos\alpha_i & \sin\alpha_i \end{bmatrix} \quad (29)$$

$$\overline{[EK]} = \frac{EA_i}{L_i}\left[\begin{array}{cccc} \cos^2\alpha_i & \cos\alpha_i \sin\alpha_i & -\cos^2\alpha_i & -\cos\alpha_i \sin\alpha_i \\ \cos\alpha_i ; \sin\alpha_i & \sin^2\alpha_i & -\cos\alpha_i \sin\alpha_i & -\sin^2\alpha_i \\ -\cos^2\alpha_i & -\cos\alpha_i \sin\alpha_i & \cos^2\alpha_i & \cos\alpha_i \sin\alpha_i \\ -\cos\alpha_i \sin\alpha_i & -\sin^2\alpha_i & \cos\alpha_i \sin\alpha_i & \sin^2\alpha_i \end{array}\right] \quad \text{Element Stiffness Matrix} \quad (30)$$

c. Summation of the element stiffness matrices to form the system stiffness matrix.

In the sketch shown, take direction from the initial to the terminal end of the member according to the arrow direction and calculate the element stiffness matrices.

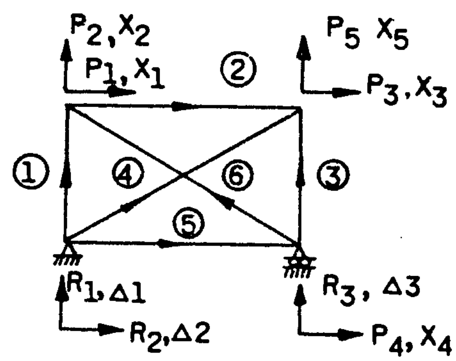

Fig B.3d

$$\left(\frac{EA_1}{L_1}\right) \begin{array}{c} \\ R_2 \\ R_1 \\ P_1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_2 & \Delta_1 & X_1 & X_2 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \end{array} ; \left(\frac{EA_2}{L_2}\right) \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_5 \end{array} \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & X_3 & X_5 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\left(\frac{EA_3}{L_3}\right) \begin{array}{c} \\ P_4 \\ R_3 \\ P_3 \\ P_5 \end{array} \begin{array}{c} \begin{array}{cccc} X_4 & \Delta_3 & X_3 & X_5 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \end{array} ; \left(\frac{EA_4}{L_4}\right) \begin{array}{c} \\ R_2 \\ R_1 \\ P_3 \\ P_5 \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_2 & \Delta_1 & X_3 & X_5 \end{array} \\ \left[\begin{array}{cccc} -.64 & 48 & -.64 & -.48 \\ -.48 & .36 & -.48 & -.36 \\ -.64 & -.48 & -.64 & .48 \\ -.48 & -.36 & .48 & .36 \end{array}\right] \end{array}$$

$$\left(\frac{EA_5}{L_5}\right) \begin{array}{c} \\ R_2 \\ R_1 \\ P_4 \\ R_3 \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_2 & \Delta_1 & X_4 & \Delta_3 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} ; \left(\frac{EA_6}{L_6}\right) \begin{array}{c} \\ P_4 \\ R_3 \\ P_1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cccc} X_4 & \Delta_3 & X_1 & X_2 \end{array} \\ \left[\begin{array}{cccc} .64 & -.48 & -.64 & .48 \\ -.48 & .36 & .48 & -.36 \\ -.64 & -.48 & .64 & -.48 \\ .48 & -.36 & -.48 & .36 \end{array}\right] \end{array}$$

The system stiffness matrix is now obtained by summing the element stiffness matrices. The addition is done by summing the coefficients having identical row and column identifications.

		x_1	x_2	x_3	x_4	x_5	Δ_1	Δ_2	Δ_3
P_1		$\left(\frac{A_2}{L_2} + .64\frac{A_6}{L_6}\right)$							
P_4		$-.48\frac{A_6}{L_6}$	$\left(\frac{A_1}{L_1} + .36\frac{A_6}{L_6}\right)$						
P_3	(E)	$-\frac{A_2}{L_2}$	0	$\left(\frac{A_2}{L_2} + .64\frac{A_4}{L_4}\right)$				Transpose	
P_4		$-.64\frac{A_6}{L_6}$	$-.48\frac{A_6}{L_6}$	- 0	$\left(\frac{A_5}{L_5} + .64\frac{A_6}{L_6}\right)$			of	
P_5		0	0	$.48\frac{A_4}{L_4}$	0	$\left(\frac{A_3}{L_3} + .36\frac{A_4}{L_4}\right)$			

R_1	0	$-\frac{A_1}{L_1}$	$-.48\frac{A_4}{L_4}$	0	$-.36\frac{A_4}{L_4}$
R_2 (E)	0	0	$-.64\frac{A_4}{L_4}$	$-\frac{A_5}{L_5}$	$-.48\frac{A_5}{L_5}$
R_3	$-.48\frac{A_6}{L_6}$	$-.36\frac{A_6}{L_6}$	0	$.48\frac{A_6}{L_6}$	$-\frac{A_3}{L_3}$

$\left(\frac{A_1}{L_1} + .36\frac{A_4}{L_4}\right)$		
$.48\frac{A_4}{L_4}$	$\left(.64\frac{A_4}{L_4} + \frac{A_5}{L_5}\right)$	
0	0	$\left(\frac{A_3}{L_3} + .36\frac{A_6}{L_6}\right)$

The matrix equations may be written as

$$\{P\} = [K]\{X\} + [K_{ER}]\{\Delta\}$$

$$\{R\} = [K_{ER}]^T\{x\} + [K_{RR}]\{\Delta\}$$

Since the $\{X\}$ matrix contains the only unknowns

$$\{X\} = [K]^{-1} [\{P\} - [K_{ER}]\{\Delta\}]$$

Up to this point we have attempted to develop the fundamental basis upon which the displacement method of structural analysis is based. The development was evolved in the following order.

1. Matrices and elementary matrix operations.

2. Simple examples illustrating the use of matrices in the solution of statically determinant problems from equilibrium concepts.

3. The basic displacement method approach involving the determination of equilibrium, stress-strain, and compatibility matrices and the combination of these to determine the system stiffness matrix.

4. Development of the system stiffness matrix directly from physical considerations and using this concept to help rationalize the development of the stiffness matrix as the summation of the element stiffnesses of the component parts.

Thus, the most important feature in developing any computer program utilizing the displacement method is in the proper formulation of element stiffness matrices. We have developed the element stiffness matrix for an axial force member. STRUDL develops additional element stiffness matrices for members comprising plane frame, plane grid, space truss and space frame systems.